

One-Step Exponential-rational Methods for the Numerical Solution of First Order Initial Value Problems

(Kaedah Eksponen-Nisbah Satu Langkah Bagi Penyelesaian Masalah Nilai Awal Peringkat Pertama Secara Berangka)

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ABSTRACT

In this study, a new class of exponential-rational methods (ERMs) for the numerical solution of first order initial value problems has been developed. Developments of third order and fourth order ERMs, as well as their corresponding local truncation error have been presented. Each ERM was found to be consistent with the differential equation and L-stable. Numerical experiments showed that the third order and fourth order ERMs generates more accurate numerical results compared with the existing rational methods in solving first order initial value problems.

Keywords: Exponential function; exponential-rational method; problem whose solution possesses singularity; rational function; rational method

ABSTRAK

Dalam kajian ini, satu kelas kaedah eksponen-nisbah untuk penyelesaian masalah nilai awal peringkat pertama secara berangka telah dibangunkan. Pembangunan kaedah eksponen-nisbah yang berperingkat ketiga dan keempat telah dibentangkan bersama dengan ralat pangkasan setempat yang sepadan. Setiap kaedah eksponen-nisbah didapati adalah tekal dengan persamaan pembezaan dan mempunyai kestabilan jenis L. Eksperimen secara berangka telah menunjukkan bahawa kaedah eksponen-nisbah berperingkat ketiga dan keempat menjana keputusan berangka yang lebih tepat berbanding dengan kaedah-kaedah nisbah yang sedia ada dalam menyelesaikan masalah nilai awal peringkat pertama.

Kata kunci: Fungsi eksponen; fungsi nisbah; kaedah eksponen-nisbah; kaedah nisbah; masalah dengan penyelesaian yang mempunyai singular

INTRODUCTION

When the solution of an initial value problem is known to possess singularity, it is appropriate to use numerical methods that are based on the local representation of a rational function of the theoretical solution $y(x)$ of the initial value problem. Several studies such as Fatunla (1986) and van Niekerk (1988) had showed that the performances of numerical methods that are based on the local representation of a polynomial were generally poor when solving problem whose solution possesses singularity. In these cases, some researches have been carried out to develop numerical methods that are based on rational function which were found to be effective in integrating problem whose solution possesses singularity.

For the sake of convenience, we have addressed these special numerical methods that are based on rational functions as rational methods. Various formulations of existing rational methods can be found in the following articles or texts: Fatunla (1982, 1986), Ikhile (2001, 2002, 2004), Lambert (1974), Lambert & Shaw (1965), Luke et al. (1975), Okosun & Ademiluyi (2007a, 2007b),

Ramos (2007), van Niekerk (1987, 1988) and Wambecq (1976).

As we have mentioned earlier, the original purpose of rational methods is to deal with problem whose solution possesses singularity. However, from the above mentioned articles, rational methods were also applied to general initial value problems whose solutions possess no singularities. These problems include stiff problems and problems with oscillatory solutions. Therefore the applicability of rational methods is not limited in solving problem whose solution possesses singularity but having greater potential in solving even more general initial value problems.

The main objective of this study was to explore the possibilities of developing new explicit rational methods which perform as effectively as the existing one or even better than the existing one. Following from this objective, we would like to find out which kind of rational methods are outstanding and which are not, because the numerical comparisons among these existing rational methods have not been carried out and we still don't know which kind of formulations are outstanding and which are not.

PRELIMINARIES

We are considering the initial value problem:

$$y' = f(x,y), y(a) = \eta, \tag{1}$$

where $y, f(x,y) \in \lambda, x \in [a,b] \subset \lambda$ and is assumed to satisfy all the conditions in order that (1) has a unique solution. The interval $[a,b]$ is divided into a number of subintervals $[x_n, x_{n+1}]$ with $x_0 = a$ and $x_n = x_0 + nh$, such that h is the step-size. Suppose that we have solved numerically the initial value problem in (1) up to a point x_n and have obtained a value y_n as an approximation of $y(x_n)$, which is the theoretical solution of (1). From Lambert (1973, 1991), assuming the localizing assumption that no previous truncation errors have been made, i.e. $y_n = y(x_n)$, we are interested in obtaining y_{n+1} as the approximation of $y(x_{n+1})$. For that purpose, we suggest an approximation to the theoretical solution $y(x_{n+1})$ of (1) given by:

$$y_{n+1} = \frac{\sum_{j=0}^k a_j h^j + c_1 e^{c_2 h}}{1 + bh}, \quad 1 + bh \neq 0, \tag{2}$$

where b, c_1, c_2 and a_j for $j = 0, 1, \dots, k$ are parameters that may contain approximations of $y(x_n)$ and higher derivatives of $y(x_n)$.

We regard (2) as one-step exponential-rational method (in brief as ERM). If the ERM has order p , then this particular ERM is called a p -ERM. From (2), we can see that ERM is a class of one-step explicit methods that are based on rational function in basis, but the numerators are combinations of polynomials and exponential functions while the denominators remain as polynomial expressions. This is a very different approach because all the existing one-step rational methods are based on conventional rational functions, where both numerators and denominators are purely polynomial expressions. From our readings, we found out that the stability conditions of a particular class of rational methods are affected by the underlying rational functions. All existing rational methods mentioned earlier were based on conventional rational functions. If these existing rational methods are applied to the scalar test problem $y' = \lambda y, y(a) = y_0, \text{Re}(\lambda) < 0$, then we can see that none of them give an exact solution to the above mentioned test problem. In other words, none of the existing rational method is exponentially-fitted. There are two advantages for a numerical method being exponentially-fitted: firstly, it returns the exact solution to the above mentioned test problem and secondly, L -stability is guaranteed (Wu 1998). In view of this, the new ERM in (2) is based on a specially designed rational function incorporated with exponential function so that the ERM is exponentially-fitted and possesses L -stability.

With the p -ERMs in (2), we associate the difference operator L defined by:

$$L[y(x);h]_{p\text{-ERM}} = y(x+h) \times (1+bh) - \sum_{j=0}^k a_j h^j - c_1 e^{c_2 h}, \quad p \geq 3, \tag{3}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $x \in [a,b] \subset \lambda$. Expanding $y(x+h)$ and exponential function $e^{c_2 h}$ as Taylor series and collecting terms in (3) gives the following general expression:

$$L[y(x);h]_{p\text{-ERM}} = C_0 h^0 + C_1 h^1 + \dots + C_k h^k + C_{k+1} h^{k+1} + C_{k+2} h^{k+2} + \dots \tag{4}$$

We note that C_i for $i = 0, 1, 2, \dots$ in (4) contain corresponding parameters that need to be determined in the derivation processes. Therefore, the order and local truncation errors of p -ERM based on (4) is defined as follows:

Definition 1.1 The difference operator (3) and the associated exponential-rational method (2) is said to be of order $p = k + 3$ if, in (4), $C_0 = C_1 = \dots = C_{k+3} = 0, C_{k+4} \neq 0$.

Definition 1.2 The local truncation error at x_{n+1} of (2) is defined to be the expression $L[y(x_n);h]_{p\text{-ERM}}$ given by (3), when $y(x_n)$ is the theoretical solution of the initial value problem (1) at a point x_n . The local truncation error of (2) is then,

$$L[y(x_n); h]_{p\text{-ERM}} = C_{k+4} h^{k+4} + O(h^{k+5}). \tag{5}$$

This article is organized as follows. Firstly, the derivations of ERMs of order 3 and 4 are presented. Next, the newly derived ERMs are analyzed in the contexts of consistency and absolute stability. Then, numerical experimentations and numerical comparisons are carried out in order to verify the validity of the new ERMs. Lastly, we end this article with a conclusion.

THIRD ORDER EXPONENTIAL-RATIONAL METHODS

The third order exponential-rational method can be obtained by taking $k = 0$ in (2) which yield the following formula:

$$y_{n+1} = \frac{a_0 + c_1 e^{c_2 h}}{1 + bh}, \quad 1 + bh \neq 0. \tag{6}$$

To determine the parameters a_0, c_1, c_2 and b , we take $k = 0$ in (3) and expand $y(x+h)$ and $e^{c_2 h}$ into series, so that the following expression is obtained by Definition 1.1:

$$L[y(x); h]_{p\text{-ERM}} = C_0 + C_1 h + C_2 h^2 + C_3 h^3 + C_4 h^4 + O(h^5), \tag{7}$$

where

$$\left\{ \begin{aligned} C_0 &= -a_0 - c_1 + y(x), C_1 = -c_1 c_2 + b y(x) + y'(x), \\ C_2 &= -\frac{1}{2} c_1 c_2^2 + b y'(x) + \frac{1}{2} y''(x), \\ C_3 &= -\frac{1}{6} c_1 c_2^3 + \frac{1}{2} b y''(x) + \frac{1}{6} y'''(x), \\ C_4 &= -\frac{1}{24} c_1 c_2^4 + \frac{1}{6} b y'''(x) + \frac{1}{24} y^{(4)}(x) \end{aligned} \right\}$$

When $y(x)$ is taken as the theoretical solution of the initial value problem (1) at a point x_n i.e. $y(x) = y(x_n)$ and letting $C_0 = C_1 = C_2 = C_3 = 0$, we obtain a system of four simultaneous equations which produces two sets of solutions:

$$\left\{ \begin{aligned} a_0 &= y_n + \frac{2(y'_n)^2 - y_n y''_n}{c_2(c_2 y_n - 2y'_n)}, b = \frac{c_2 y'_n - y''_n}{2y'_n - c_2 y_n}, \\ c_1 &= \frac{y_n y''_n - 2(y'_n)^2}{c_2(c_2 y_n - 2y'_n)}, c_2 = \frac{y_n y'''_n - 3y'_n y''_n - U}{2(y_n y''_n - 2(y'_n)^2)} \end{aligned} \right\} \quad (8)$$

and

$$\left\{ \begin{aligned} a_0 &= y_n + \frac{2(y'_n)^2 - y_n y''_n}{c_2(c_2 y_n - 2y'_n)}, b = \frac{c_2 y'_n - y''_n}{2y'_n - c_2 y_n}, \\ c_1 &= \frac{y_n y''_n - 2(y'_n)^2}{c_2(c_2 y_n - 2y'_n)}, c_2 = \frac{y_n y'''_n - 3y'_n y''_n - U}{2(y_n y''_n - 2(y'_n)^2)} \end{aligned} \right\} \quad (9)$$

where $y_n = y(x_n)$ and $y_n^{(m)} = y^{(m)}(x_n)$ for $m = 1, 2, 3$ by the localizing assumption. We note that $U = \sqrt{(3y'_n y''_n - y_n y'''_n)^2 - 4(-2(y'_n)^2 + y_n y''_n)(-3(y''_n)^2 + 2y'_n y'''_n)}$.

We indicate (6) based on (8) as 3-ERM(1) while (6) based on (9) as 3-ERM(2). From Definition 1.2, the local truncation error (in brief as LTE) of 3-ERM(1) and 3-ERM(2) are:

$$\text{LTE}_{3\text{-ERM}(1)} = \frac{h^5}{192} \left(\frac{-2(3y'_n y''_n - y_n y'''_n + U)^3}{(2(y'_n)^2 - y_n y''_n)(8(y'_n)^3 - 7y_n y'_n y''_n + y_n^2 y'''_n - y_n^2 U)} - \frac{32y_n^3((y'_n)^3 y''_n - 2y_n(y''_n)^3 + y_n y'_n y'''_n - y'_n U)}{8(y'_n)^3 - 7y_n y'_n y''_n + y_n^2 y'''_n - y_n^2 U} + 8y_n^{(5)} \right) + O(h^5) \quad (10)$$

and

$$\text{LTE}_{3\text{-ERM}(2)} = \frac{h^5}{192} \left(\frac{2(-3y'_n y''_n + y_n y'''_n + U)^3}{(2(y'_n)^2 - y_n y''_n)(8(y'_n)^3 - 7y_n y'_n y''_n + y_n^2 y'''_n + y_n U)} - \frac{32y_n^3((y'_n)^3 y''_n - 2y_n(y''_n)^3 + y_n y'_n y'''_n + y'_n U)}{8(y'_n)^3 - 7y_n y'_n y''_n + y_n^2 y'''_n + y_n U} + 8y_n^{(5)} \right) + O(h^5) \quad (11)$$

respectively, where $y_n = y(x_n)$ and $y_n^{(m)} = y^{(m)}(x_n)$ for $m = 1, 2, 3, 4$ by the localizing assumption.

FOURTH ORDER EXPONENTIAL-RATIONAL METHODS

The fourth order exponential-rational method can be obtained by taking $k = 1$ in (2) which yield the following formula:

$$y_{n+1} = \frac{a_0 + a_1 h + c_1 e^{c_2 h}}{1 + bh}, \quad 1 + bh \neq 0. \quad (12)$$

To determine the parameters a_0, a_1, c_1, c_2 and b , we take $k = 1$ in (3) and expand $y(x+h)$ and $e^{c_2 h}$ into series, so that the following expression is obtained:

$$L[y(x);h]_{p\text{-ERM}} = C_0 + C_1 h + C_2 h^2 + C_3 h^3 + C_4 h^4 + C_5 h^5 + O(h^6), \quad (13)$$

where

$$\begin{aligned} \{C_0 &= -a_0 - c_1 y(x), C_1 = -a_1 - c_1 c_2 + b y'(x) + y'(x), \\ C_2 &= -\frac{1}{2} c_1 c_2^2 + b y'(x) + b y'(x) + \frac{1}{2} y''(x), \\ C_3 &= -\frac{1}{6} c_1 c_2^3 + \frac{1}{2} b y''(x) + \frac{1}{6} y'''(x), \\ C_4 &= -\frac{1}{24} c_1 c_2^4 + \frac{1}{6} b y'''(x) + \frac{1}{24} y^{(4)}(x), \\ C_5 &= -\frac{1}{120} c_1 c_2^5 + \frac{1}{24} b y^{(4)}(x) + \frac{1}{120} y^{(5)}(x)\}. \end{aligned}$$

When $y(x)$ is taken as the theoretical solution of the initial value problem (1) at a point x_n i.e. $y(x) = y(x_n)$, and letting $C_0 = C_1 = C_2 = C_3 = C_4 = 0$, we obtain a system of five simultaneous equations which produces two sets of solutions:

$$\left\{ \begin{aligned} a_0 &= y_n + \frac{3(y''_n)^2 - 2y'_n y'''_n}{c_2^2(2c_2 y'_n - 3y''_n)}, a_1 = y'_n + \frac{y_n(c_2 y''_n - y'''_n)}{3y''_n - 2c_2 y'_n} + \frac{3(y''_n)^2 - 2y'_n y'''_n}{c_2(2c_2 y'_n - 3y''_n)}, \\ b &= \frac{c_2 y''_n - y'''_n}{3y''_n - 2c_2 y'_n}, c_1 = \frac{2y'_n y''_n - 3(y''_n)^2}{c_2^2(2c_2 y'_n - 3y''_n)}, c_2 = \frac{2y'_n y^{(4)}_n - 4y''_n y'''_n - V}{2(2y'_n y''_n - 3(y''_n)^2)} \end{aligned} \right\} \quad (14)$$

and

$$\left\{ \begin{aligned} a_0 &= y_n + \frac{3(y''_n)^2 - 2y'_n y'''_n}{c_2^2(2c_2 y'_n - 3y''_n)}, a_1 = y'_n + \frac{y_n(c_2 y''_n - y'''_n)}{3y''_n - 2c_2 y'_n} + \frac{3(y''_n)^2 - 2y'_n y'''_n}{c_2(2c_2 y'_n - 3y''_n)}, \\ b &= \frac{c_2 y''_n - y'''_n}{3y''_n - 2c_2 y'_n}, c_1 = \frac{2y'_n y''_n - 3(y''_n)^2}{c_2^2(2c_2 y'_n - 3y''_n)}, c_2 = \frac{2y'_n y^{(4)}_n - 4y''_n y'''_n - V_2}{2(2y'_n y''_n - 3(y''_n)^2)} \end{aligned} \right\} \quad (15)$$

where $y_n = y(x_n)$ and $y_n^{(m)} = y^{(m)}(x_n)$ for $m = 1, 2, 3, 4$ by the localizing assumption. We note that

$$V = \sqrt{(4y''_n y'''_n - 2y'_n y^{(4)}_n)^2 - 4(-3(y''_n)^2 + 2y'_n y'''_n)(-4(y''_n)^2 + 3y''_n y^{(4)}_n)}$$

We indicate (12) based on (14) as 4-ERM(1) while (12) based on (15) as 4-ERM(2). From Definition 1.2, the LTE of 4-ERM(1) and 4-ERM(2) are

$$\text{LTE}_{4\text{-ERM}(1)} = \frac{h^5}{192} \left(\frac{-(2y''_n y'''_n - y'_n y^{(4)}_n + \frac{1}{2} V_1)^3}{(3(y'_n)^2 - 2y'_n y''_n)(9(y'_n)^3 - 10y_n y'_n y''_n + 2(y'_n)^2 y_n^{(3)} - y'_n V_1)} - \frac{5y_n^3((y'_n)^3 y''_n - 2y'_n(y''_n)^3 + y_n y'_n y'''_n - \frac{1}{2} V_1)}{9(y'_n)^3 - 10y_n y'_n y''_n + 2(y'_n)^2 y_n^{(3)} - y_n V_1} + y_n^{(5)} \right) + O(h^6) \quad (16)$$

and

$$\text{LTE}_{4\text{-ERM}(2)} = \frac{h^5}{120} \left(\frac{(-2y''_n y'''_n + y'_n y^{(4)}_n + \frac{1}{2} V_2)^3}{(3(y'_n)^2 - 2y'_n y''_n)(9(y'_n)^3 - 10y_n y'_n y''_n + 2(y'_n)^2 y_n^{(3)} + y'_n V_2)} - \frac{5y_n^3((y'_n)^3 y''_n - 2y'_n(y''_n)^3 + y_n y'_n y'''_n + \frac{1}{2} V_2)}{9(y'_n)^3 - 10y_n y'_n y''_n + 2(y'_n)^2 y_n^{(3)} + y_n V_2} + y_n^{(5)} \right) + O(h^6), \quad (17)$$

respectively, where $y_n = y(x_n)$ and $y_n^{(m)} = y^{(m)}(x_n)$ for $m = 1, 2, 3, 4, 5$ by the localizing assumption.

CONSISTENCY AND ABSOLUTE STABILITY
OF 3-ERMS AND 4-ERMS

We now show that p -ERM(1) and p -ERM(2) for $p = 3, 4$ are consistent with the differential in (1) by the following definition.

Definition 4.1 The ERM in (2) is said to be consistent if (5) satisfy,

$$\lim_{\substack{h \rightarrow 0 \\ x = a + nh}} \frac{1}{h} L[y(x_n); h]_{p\text{-ERM}} = 0. \tag{18}$$

From Definition 1.2, $L[y(x_n); h]_{p\text{-ERM}}$ is essentially the local truncation error for a p -th order ERM. On using *Mathematica 8.0*, it can be shown that the local truncation errors given in (10), (11), (16) and (17) satisfy the condition in (18), which directly implies that p -ERM(1) and p -ERM(2) for $p = 3, 4$ are consistent with the differential in (1).

The absolute stability analysis of a p -ERM(1) or p -ERM(2) can be obtained easily by applying the ERMS proposed in this paper to the scalar test problem:

$$y' = \lambda y, y(a) = y_0, \text{Re}(\lambda) < 0. \tag{19}$$

It can be shown that the applications of these proposed ERMS to the test problem in (19) resulted in the following difference equation:

$$y_{n+1} = R(z)y_n, z = h\lambda. \tag{20}$$

We note that $R(z)$ is the stability function for any of the p -ERM proposed in this paper. Clearly $y_n \rightarrow 0$ as $n \rightarrow \infty$ if and only if,

$$|R(z)| < 1. \tag{21}$$

A p -ERM is absolutely stable for those values of z for which condition in (21) holds. The region of absolute stability of a p -ERM is defined as $\{z \in \mathbb{C} : |R(z)| \leq 1\}$ or the set of points in the complex plane such that the approximated solution remains bounded after many steps of computations (Butcher 2003).

On using *Mathematica 8.0*, it can be shown that the stability functions for p -ERM(1) and p -ERM(2) for $p = 3, 4$ are identical, i.e.

$$R(z) = R(z)_{\text{ERM}} = e^z. \tag{22}$$

Hence, the stability regions for these four ERMS are also identical as illustrated in Figure 1.

By using (20) and (22), applications of all ERMS presented in this paper to the test problem in (19) resulted in the following difference equation:

$$y_{n+1} = e^{h\lambda} y_n. \tag{23}$$

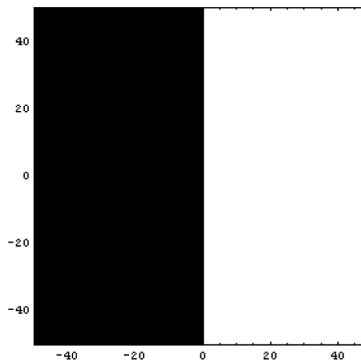


FIGURE 1. Stability region of p -ERM(1) and p -ERM(2) for $p = 3, 4$

We note that $h\lambda$ is not represented by z for the ease of explanation. We can write (23) as:

$$y_n = (e^{h\lambda})^n y_0. \tag{24}$$

Since $(e^{h\lambda})^n \rightarrow 0$ as $n \rightarrow \infty$ for all $z = h\lambda$ with $\text{Re}(\lambda) < 0$, we have $y_n \rightarrow 0$ as $n \rightarrow \infty$. Consequently, p -ERM(1) and p -ERM(2) for $p = 3, 4$ are A -stable (Wu 1998). For the numerical solutions of stiff problems, A -stable method is desirable and sometimes L -stable method is more preferable especially in solving excessive stiff problems.

Definition 4.2 (Lambert 1991) A numerical method is said to be L -stable if it is A -stable and in addition, when applied to the scalar test problem (19), it yields (20), where $|R(z)| \rightarrow 0$ as $\text{Re}(z) \rightarrow -\infty$.

It is easy to see that p -ERM(1) and p -ERM(2) for $p = 3, 4$ are L -stable because it is readily deduced that $|R(z)| \rightarrow 0$ as $\text{Re}(z) \rightarrow -\infty$ from (22).

NUMERICAL EXPERIMENTS AND COMPARISONS

It is very obvious that ERMS proposed in this paper will produce numerical solutions which are complex numbers due to the square root evaluations. To retrieve numerical solutions which are only real numbers, we shall only consider the real parts of the resulting complex values and ignore the imaginary parts of the complex values. It can be shown from computations, that the imaginary parts of the complex values are very small. The numerical experiments below will show that this numerical strategy taken does not affect the computational process and also yield better and satisfying numerical results.

Some test problems are used to check the accuracy of the proposed ERMS using different number of integration steps. We present the maximum absolute errors over the integration interval given by $\max_{0 \leq n \leq N} \{|y(x_n) - y_n|\}$ where N is the number of integration steps. We note that $y(x_n)$ and y_n represent the exact solution and numerical solution of a test problem at point x_n , respectively. Step-size is chosen to fulfill stability and accuracy requirements. The numerical results obtained from the proposed ERMS are compared

with the numerical results obtained from existing third order and fourth order rational methods mentioned in Ikhile (2001), Lambert & Shaw (1965) and van Niekerk (1987, 1988). We note that the third order methods of Lambert & Shaw (1965) and van Niekerk (1988) are identical, while the fourth order methods of van Niekerk (1987, 1988) are identical.

Problem 1 (Fatunla 1976)

$$y'(x) = -2xy(x) + 4x, y(0) = 3, x \in [0, 0.5].$$

The theoretical solution is $y(x) = e^{-x^2} + 2$.

Problem 2 (Ramos 2007)

$$y'(x) = -100y(x) + 99e^{2x}, y(0) = 0, x \in [0, 10].$$

The theoretical solution is $y(x) = \frac{33}{34}(e^{2x} - e^{-100})$.

Problem 3 (Fatunla 1982)

$$y'(x) = -2000e^{-200x} + 9e^{-x}, y(0) = 10, x \in [0, 10].$$

The theoretical solution is $y(x) = 10 - 10e^{-x} - xe^{-x} + 10e^{-200x}$.

Problem 4 (Ramos 2007)

$$y'_1(x) = -1002y_1(x) + 1000y_2(x)^2, y_1(0) = 1, x \in [0, 1];$$

$$y'_2(x) = y_1(x) - y_2(x)(1 + y_2(x)), y_2(0) = 1, x \in [0, 1].$$

The theoretical solutions are $y_1(x) = e^{-2x}$ and $y_2(x) = e^{-x}$.

Problem 5 (Yaakub & Evans 2003)

$$y''(x) + 101y'(x) + 100y(x) = 0, y(0) = 1.01,$$

$$y'(0) = -2, x \in [0, 1].$$

The theoretical solution is $y(x) = 0.01e^{-100x} + e^{-x}$. *Problem 2* can be written as a system, i.e.

$$y'_1(x) = y_2(x), y_1(0) = 1.01, x \in [0, 10];$$

$$y'_2(x) = -100y_1(x) - 101y_2(x), y_2(0) = -2, .$$

The theoretical solutions are $y_1(x) = 0.01e^{-100x} + e^{-x}$ and $y_2(x) = -e^{-100x} - e^{-x}$.

Problem 6 (Ramos 2007)

$$y'(x) = 1 + y(x)^2, y(0) = 1, x \in [0, 0.8].$$

The theoretical solution is $y(x) = \tan(x + \pi/4)$.

Problem 6 is an example of problem whose solution possesses singularity. From the theoretical solution, notice that the solution becomes unbounded in the neighbourhood of the singularity at $x = \pi/4 \approx 0.785398163367448$.

From Table 1, we can see that the third order rational method of Ikhile (2001) generated the least accurate numerical results compared with the remaining third order rational methods that are found to have comparable accuracy in solving *Problem 1*. However, 4-ERM(1) and 4-ERM(2) turn out to have an excellent accuracy compared with other existing fourth order rational methods for any number of integration steps (Table 2).

The results from Table 3 shows that 3-ERM(2) generated the most accurate numerical results in solving *Problem 2*, followed by 3-ERM(1) and the method by Ikhile (2001) was found to have a comparable accuracy. In other words, 3-ERM(1), 3-ERM(2) and the method by Ikhile (2001) suggest faster convergence compared with the methods by Lambert and Shaw (1965), van Niekerk (1987, 1988) in solving *Problem 2* which is a stiff problem. Meanwhile, results from Table 4 shows that 4-ERM(1) is the most accurate fourth order method compared with

TABLE 1. Maximum absolute errors for various third order methods with respect to the number of steps (*Problem 1*)

<i>N</i>	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	3-ERM(1)	3-ERM(2)
16	8.35079(-06)	3.35810(-06)	8.35079(-06)	1.39243(-03)	4.99376(-06)	4.99376(-06)
32	1.06661(-06)	4.27061(-07)	1.06661(-06)	3.56759(-04)	6.30791(-07)	6.30791(-07)
64	1.34848(-07)	5.38534(-08)	1.34848(-07)	9.05271(-05)	7.92800(-08)	7.92800(-08)

TABLE 2. Maximum absolute errors for various fourth order methods with respect to the number of steps (*Problem 1*)

<i>N</i>	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	4-ERM(1)	4-ERM(2)
16	4.76682(-07)	4.30940(-08)	4.30940(-08)	9.76086(-04)	1.49641(-09)	1.49641(-09)
32	2.97999(-08)	2.74378(-09)	2.74378(-09)	2.44111(-04)	7.84479(-11)	7.84439(-11)
64	1.86261(-09)	1.73048(-10)	1.73048(-10)	6.10333(-05)	4.44356(-12)	4.44489(-12)

TABLE 3. Maximum absolute errors for various third order methods with respect to the number of steps (*Problem 2*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	3-ERM(1)	3-ERM(2)
640	6.23940(+00)	6.25559(+00)	6.23940(+00)	1.95242(-02)	1.25561(-02)	7.65251(-05)
1280	6.89217(-01)	6.93903(-01)	6.89217(-01)	1.08057(-03)	1.51587(-03)	8.91571(-06)
2560	7.37724(-02)	7.40682(-02)	7.37724(-02)	7.41153(-05)	1.85829(-04)	8.91571(-06)

TABLE 4. Maximum absolute errors for various fourth order methods with respect to the number of steps (*Problem 2*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	4-ERM(1)	4-ERM(2)
640	3.93344(-02)	2.67154(-02)	2.67154(-02)	2.67154(-02)	7.88357(-06)	2.82945(-03)
1280	1.65933(-03)	1.11371(-03)	1.11371(-03)	1.11371(-03)	3.46331(-07)	8.98964(+01)
2560	8.65459(-05)	5.79357(-05)	5.79357(-05)	5.79357(-05)	1.19209(-07)	1.33235(-02)

other existing methods that are found to have comparable accuracy. However, 4-ERM(2) is found to be less accurate for $N = 1280$ and $N = 2560$.

Problem 3 is also a stiff problem, but very much ‘stiffer’ than *Problem 2*. From Table 5, it is obvious that 3-ERM(1) and 3-ERM(2) converge to the theoretical solution faster than other existing third order rational methods. The same pattern emerges in Table 6 where 4-ERM(1) and 4-ERM(2) also converge to the exact solution faster than other existing fourth order rational methods. The observations from Table 5 and 6 suggest that third order and fourth order rational methods of Ikhile (2001) converge slowly to the exact solution especially for $N = 100000$.

Problem 4 is a stiff system. The results from Tables 7 and 8 show that 3-ERM(1) and 3-ERM(2) are more accurate than

other existing third order rational method in computing the components $y_1(x)$ and $y_2(x)$ for *Problem 4*. We wish to highlight the tremendous achievement of 3-ERM(1) for $N = 160$ as shown in Tables 7 and 8. This same pattern can also be seen when 4-ERM(1) and 4-ERM(2) achieved greater accuracy for $N = 160$ as shown in Tables 9 and 10. In view of this, we can say that 3-ERM(1), 4-ERM(1) and 4-ERM(2) possess the potential to achieve high accuracy with a smaller number of integration steps in solving *Problem 4*.

We note that third order rational methods of Lambert and Shaw (1965), van Niekerk (1987) and van Niekerk (1988) are found to have comparable accuracy for $N = 320$ and $N = 640$. Third order methods of Ikhile (2001), 3-ERM(1) and 3-ERM(2) are found to have comparable accuracy for $N = 640$. On the other hand, all fourth order rational methods presented in Tables 9 and 10 are found to have comparable accuracy for $N = 640$.

TABLE 5. Maximum absolute errors for various third order methods with respect to the number of steps (*Problem 3*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	3-ERM(1)	3-ERM(2)
100	7.08987(+01)	1.51505(+00)	7.08987(+01)	4.71235(+00)	8.05125(-01)	8.05075(-01)
1000	7.48249(-01)	3.57558(-01)	7.48249(-01)	6.24419(-02)	2.36491(-02)	1.43271(-01)
10000	1.06282(-03)	1.44188(-03)	1.06282(-03)	1.34363(-03)	2.76633(-05)	1.93116(-04)
100000	1.10728(-06)	1.89317(-06)	1.10728(-06)	1.44295(-05)	2.86579(-08)	6.40614(-07)

TABLE 6. Maximum absolute errors for various fourth order methods with respect to the number of steps (*Problem 3*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	4-ERM(1)	4-ERM(2)
100	4.12223(+02)	5.01565(+00)	5.01565(+00)	4.99044(+00)	2.35541(-02)	2.35541(-02)
1000	2.80158(-01)	8.19748(-02)	8.19748(-02)	8.10986(-02)	1.71212(-03)	1.98398(-05)
10000	3.27884(-05)	1.83181(-05)	1.83181(-05)	6.42436(-04)	4.49223(-07)	1.18550(-09)
100000	3.33261(-09)	1.98984(-09)	1.98984(-09)	1.26886(-05)	5.23190(-11)	4.18554(-13)

TABLE 7. Maximum absolute errors for various third order methods with respect to the number of steps ($y_1(x)$) (*Problem 4*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	3-ERM(1)	3-ERM(2)
160	2.19212(+02)	3.17981(-01)	2.19212(+02)	8.23205(-03)	5.19877(-05)	3.96606(+01)
320	2.90442(-05)	3.84679(-05)	2.90442(-05)	1.34220(-03)	1.99991(-06)	3.28414(-06)
640	2.01537(-11)	2.01373(-11)	2.01537(-11)	4.49640(-15)	4.21885(-15)	4.10783(-15)

TABLE 8. Maximum absolute errors for various third order methods with respect to the number of steps ($y_2(x)$) (*Problem 4*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	3-ERM(1)	3-ERM(2)
160	2.18514(-01)	5.06153(-04)	2.18514(-01)	5.28030(-03)	3.14264(-05)	4.72343(-02)
320	2.16581(-06)	2.16300(-06)	2.16581(-06)	7.00650(-05)	1.86383(-07)	6.49753(-07)
640	1.96714(-11)	1.96536(-11)	1.96714(-11)	4.10783(-15)	2.77556(-15)	2.33147(-15)

TABLE 9. Maximum absolute errors for various fourth order methods with respect to the number of steps ($y_1(x)$) (*Problem 4*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	4-ERM(1)	4-ERM(2)
160	2.69791(+05)	1.00469(-02)	1.00469(-02)	3.75787(-02)	1.21330(-07)	4.94472(-07)
320	1.28050(-08)	1.19347(-08)	1.19347(-08)	9.11811(-05)	2.68292(-10)	1.31970(-09)
640	6.93889(-15)	2.66454(-15)	2.66454(-15)	4.49640(-15)	2.77556(-15)	3.21965(-15)

TABLE 10. Maximum absolute errors for various fourth order methods with respect to the number of steps ($y_2(x)$) (*Problem 4*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	4-ERM(1)	4-ERM(2)
160	2.69781(+02)	1.75091(-04)	1.75091(-04)	3.79099(-03)	3.40338(-08)	1.63828(-07)
320	1.27173(-10)	5.95530(-11)	5.95530(-11)	5.10325(-06)	1.52687(-11)	7.94500(-11)
640	5.44009(-15)	3.05311(-15)	3.05311(-15)	4.10783(-15)	2.22045(-15)	2.10942(-15)

Problem 5 is a stiff system arises from the reduction of a second order initial value problem to a system of coupled first order differential equations. From Table 11, it has been shown that 3-ERM(1), third order rational methods of Lambert and Shaw (1965), van Niekerk (1988) and Ikhile (2001) are found to have comparable accuracy except for $N = 1280$, while 3-ERM(2) and third order method of van Niekerk (1987) are found to have comparable accuracy for any number of integration steps. From Table 12, it is obvious that 4-ERM(1) is the most accurate fourth order method while 4-ERM(2) and other existing fourth order rational methods have comparable accuracy in solving *Problem 5*, for any number of integration steps.

The results from Table 13 shows that 3-ERM(1) and 3-ERM(2) are not suitable to solve *Problem 6*, which is a problem whose solution possesses singularity. We can see that 3-ERM(1) and 3-ERM(2) are less accurate compared with other existing third order rational methods. Meanwhile, results from Table 14 shows that 4-ERM(2) is the least accurate fourth order method in comparison, which

suggests that 4-ERM(2) is not suitable to solve *Problem 6*. However, 4-ERM(1) and other existing fourth order rational methods are found to have comparable accuracy for $N = 16$ and $N = 32$. The observations from Tables 13 and 14 also revealed that the third order and fourth order rational methods of Ikhile (2001) are the most suitable methods in solving problem whose solution possesses singularity because they yield more accurate numerical results.

CONCLUSION

In this article, we have presented a new class of ERMs which are explicit one-step methods that are based on rational functions. The general formulation of ERM is given in (2) while the order condition and local truncation error for an ERM are explained in Definition 1.1 and Definition 1.2.

Four examples of ERMs have been introduced i.e. 3-ERM(1), 3-ERM(2), 4-ERM(1) and 4-ERM(2). From the process of derivations, readers must have noticed that the parameters b , c_1 , c_2 and a_j for a p -ERM are not unique, not

TABLE 11. Maximum absolute errors for various third order methods with respect to the number of steps (*Problem 5*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	3-ERM(1)	3-ERM(2)
1280	2.91323(-05)	1.67276(-04)	2.91323(-05)	2.15408(-05)	7.46251(-04)	1.68219(-04)
2560	3.12721(-06)	1.56050(-05)	3.12721(-06)	3.18139(-06)	3.31054(-06)	2.48349(-05)
5120	3.67925(-07)	1.24983(-06)	3.67925(-07)	4.38761(-07)	1.96674(-07)	1.16168(-06)

TABLE 12. Maximum absolute errors for various fourth order methods with respect to the number of steps (*Problem 5*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	4-ERM(1)	4-ERM(2)
1280	3.94471(-06)	2.07593(-05)	2.07593(-05)	1.05937(-05)	1.22646(-08)	1.38125(-05)
2560	2.08414(-07)	1.17827(-06)	1.17827(-06)	6.88079(-07)	9.47332(-10)	7.76176(-07)
5120	1.21032(-08)	7.07248(-08)	7.07248(-08)	4.39613(-08)	5.50824(-10)	5.43818(-08)

TABLE 13. Maximum absolute errors for various third order methods with respect to the number of steps (*Problem 6*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	3-ERM(1)	3-ERM(2)
16	2.39514(-02)	2.10235(-01)	2.39514(-02)	5.20857(-04)	3.93191(+00)	1.51616(-01)
32	5.73126(-03)	5.01590(-02)	5.73126(-03)	6.22138(-05)	5.87270(+00)	3.47699(-02)
64	1.72803(-02)	2.15491(-01)	1.72803(-02)	9.67085(-05)	4.01475(+00)	1.52879(-01)

TABLE 14. Maximum absolute errors for various fourth order methods with respect to the number of steps (*Problem 6*)

N	Lambert and Shaw (1965)	van Niekerk (1987)	van Niekerk (1988)	Ikhile (2001)	4-ERM(1)	4-ERM(2)
16	7.41608(-04)	5.20857(-04)	5.20857(-04)	5.20857(-04)	5.73593(-04)	3.81519(+00)
32	8.79923(-05)	6.22138(-05)	6.22138(-05)	6.22138(-05)	6.84152(-05)	1.39082(+01)
64	1.35069(-04)	9.67086(-05)	9.67086(-05)	9.67085(-05)	2.37726(-03)	1.41133(+00)

only for $p = 3, 4$ but also for any value of p . In other words, a p -th order ERM is not unique but two different methods which share the same order of accuracy i.e. p -ERM(1) and p -ERM(2). Currently, we are studying the strategy on determining which p -ERM (p -ERM(1) or p -ERM(2)) yields the most accurate results and returns only these results. Another on-going investigation is to generalize the parameters b , c_1 , c_2 and a_j for any order of accuracy. The ERMs proposed in this article are said to be consistent by Definition 4.1. Absolute stability analysis showed that the proposed ERMs are L -stable. We shall prove that all ERMs are L -stable as well as convergence regardless of the order of accuracy in a future study.

We have chosen some test problems to evaluate the effectiveness of ERMs and other existing rational methods in terms of numerical accuracy. Most of the time, ERMs generated more accurate numerical results compared with existing rational methods in solving non-stiff problem (*Problem 1*) as well as stiff problems (*Problems 2, 3, 4* and *5*). Therefore, ERMs are suitable for general initial

value problems whose solutions possess no singularity. However, ERMs did not perform as good as existing rational methods when solving problem whose solution possesses singularity. Therefore, we can claim that ERMs are not suitable for problems whose solutions possess singularities. From these numerical experimentations, we can conclude that the capability of ERMs in solving problems whose solutions possess singularities is less obvious but in return, ERMs are more reliable in solving general initial value problems especially stiff problems.

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